# SECONDARY CHARACTERISTIC CLASSES OF SURFACE BUNDLES

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ABSTRACT. The Miller-Morita-Mumford classes associate to an oriented surface bundle $E \to B$ a class $\kappa_i(E) \in H^{2i}(B;\mathbb{Z})$. In this note we define for each prime $p$ and each integer $i \geq 1$ a secondary characteristic class $\lambda_i(E) \in H^{2i(p-1)-2}(B;\mathbb{Z})/\mathbb{Z}\kappa_{i(p-1)-1}$. The mod $p$ reduction $\lambda_i(E) \in H^*(B;\mathbb{F}_p)$ has zero indeterminacy and satisfies $p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B;\mathbb{Z}/p^2)$.

## 1. Introduction and statement of results

Recall that any bundle $\pi : E \to B$ of oriented surfaces with finite dimensional base $B$ has an embedding $j : E \to B \times \mathbb{R}^{N+2}$ over $B$. For $N$ large, $j$ is unique up to isotopy. A choice of embedding $j$ induces a transfer map

$$B_+ \wedge S^{N+2} \xrightarrow{\pi_!} \mathrm{Th}(\nu j)$$

The embedding $j : E \to B \times \mathbb{R}^{N+2}$ also induces classifying maps

$$\begin{tikzcd} T_\pi E \arrow[r, "\mathrm{cl}(T_\pi E)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2 \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/\mathrm{SO}(N) \times \mathrm{SO}(2) \end{tikzcd}$$

and

$$\begin{tikzcd} \nu j \arrow[r, "\mathrm{cl}(\nu j)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/\mathrm{SO}(N) \times \mathrm{SO}(2) \end{tikzcd}$$

For brevity, write $U = U_N = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2$ and $U^\perp = U_N^\perp = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N$. We get the composition

$$\alpha = \mathrm{Th}(\mathrm{cl}(\nu j)) \circ \pi_! : B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp)$$

Recall that there is a Thom class $\lambda_{U^\perp} \in H^N(\mathrm{Th}(U^\perp), *; \mathbb{Z})$ and that we have $H^{N+*}(\mathrm{Th}(U^\perp), *; \mathbb{Z}) = \mathbb{Z}[e(U)].\lambda_{U^\perp}$ for $* < N$. The definition of the $\kappa$-classes is

$$\kappa_i E = \alpha^*(e(U)^{i+1}.\lambda_{U^\perp}) = \pi_!^*(e(T_\pi E)^{i+1}.\lambda_{\nu j}) \in H^{2i}(B;\mathbb{Z})$$

In this paper we define secondary characteristic classes of surface bundles. The definition involves Toda brackets. In section 2 we recall some generalities about Toda brackets. By a surface bundle we shall mean a fibre bundle with closed oriented smooth two-dimensional fibres.

**Lemma 1.1.** *Let $p$ be a prime, and let $\mathcal{P}^i$ denote the Steenrod power operation. When $p = 2$, write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$. Given a surface bundle $\pi : E \to B$, let $\alpha : B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp)$ be as before and let $\lambda : \mathrm{Th}(U_N^\perp) \to K(\mathbb{Z}, N)$ be the Thom class. Then the Toda bracket*

$$\{\beta\mathcal{P}^i, \lambda, \alpha\} \subseteq H^{2i(p-1)-2+N}(B_+ \wedge S^{N+2}; \mathbb{Z}) = H^{2i(p-1)-2}(B; \mathbb{Z})$$

*is defined with indeterminacy $\mathbb{Z}\kappa_{i(p-1)-1}$.*

**Definition 1.2.** With notation as in Lemma 1.1 define

$$\lambda_i(E) = (-1)^i\{\beta\mathcal{P}^i, \lambda, \alpha\} \in H^{2i(p-1)-2}(B; \mathbb{Z})/\mathbb{Z}\kappa_{i(p-1)-1}$$

**Theorem 1.3.** *The mod $p$ reduction $\lambda_i(E) \in H^*(B; \mathbb{F}_p)$ has zero indeterminacy and satisfies*

$$p\lambda_i(E) = \kappa_{i(p-1)-1} \in H^*(B; \mathbb{Z}/p^2)$$

*More generally we have the following in integral cohomology*

$$\kappa_{i(p-1)-1} \in p\lambda_i(E)$$

**Theorem 1.4.** *(i) If $\pi : E \to B$ and $\pi' : E' \to B$ are surface bundles, then*

$$\lambda_i(E \amalg E') = \lambda_i(E) + \lambda_i(E')$$

*(ii) If $\pi : E \to B$ is a surface bundles and $\pi' : E' \to B$ is obtained from $E$ by fibrewise surgery, then*

$$\lambda_i E = \lambda_i E'$$

*(iii) If $\pi : E \to B$ and $\pi' : E' \to B$ are bundles of compact, non-closed surfaces with $\partial E = S^1 \times B = \partial E'$, then*

$$\lambda_i(E \cup_{S^1\times B} E') = \lambda_i(E \cup_{S^1\times B} (D^2 \times B)) + \lambda_i(E' \cup_{S^1\times B} (D^2 \times B))$$

As an application of secondary classes we prove the following strengthening of a theorem of [GMT]:

**Theorem 1.5.** *Let $p$ be a prime and $s \geq 1$. Then the reduction of $\kappa_{ps(p-1)-1}$ mod $p^2$ vanishes:*

$$\kappa_{ps(p-1)-1} = 0 \in H^*(B; \mathbb{Z}/p^2)$$

Theorem 1.5 proves part of the following conjecture.

**Conjecture 1.6.** *Let $s \geq 1$ and $v \geq 0$. Then*

$$\kappa_{p^v s(p-1)-1} = 0 \in H^*(B; \mathbb{Z}/p^{v+1})$$

If the conjecture is true, then $\kappa_{p^v s(p-1)-1}$ can be divided by $p^{v+1}$. In [GMT] we prove that this holds modulo torsion. It is also proved in [GMT] that the statement of Conjecture 1.6 is best possible in the sense that if $s \not\equiv 0 \pmod p$, then $\kappa_{p^v s(p-1)-1} \neq 0 \in H^*(B; \mathbb{Z}/p^{v+2})$. I hope to return to Conjecture 1.6 at a later time.

## 2. Secondary composition

We recall the definition of secondary compositions (Toda brackets). For further details see [Toda].

All spaces and maps are pointed. The reduced suspension $SX$ is regarded as the pushout of $X \wedge [-1,0] \longleftarrow X \longrightarrow X \wedge [0,1]$ where $-1 \in [-1,0]$ and $1 \in [0,1]$ are the basepoints. Thus, two nullhomotopies $F : X \wedge [-1,0] \to Y$ and $G : X \wedge [0,1] \to Y$ induce a map $G - F : SX \to Y$.

For a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $g \circ f \simeq 0$ and $h \circ g \simeq 0$, a choice of null-homotopies $F : g \circ f \simeq 0$ and $G : h \circ g \simeq 0$ determines a map

$$h \circ F - G \circ (f \wedge [-1,0]) : SX \to W$$

We define the *secondary composition* to be the subset $\{h, g, f\} \subseteq [SX, W]$ of homotopy classes of maps obtained in this fashion, as $F, G$ ranges over all null-homotopies.

Recall that $[SX, W] = [X, \Omega W]$ is a group.

**Lemma 2.1.** *$\{h, g, f\}$ depends only on the homotopy classes of $h$, $g$, and $f$. If $\{h, g, f\}$ is defined, then it gives a unique element in the double coset,*

$$\{h, g, f\} \in h \circ [SX, Z] \setminus [SX, W]/[SY, W] \circ Sf$$

*If $[SX, W]$ is abelian, then*

$$\{h, g, f\} \in [SX, W]/\big(h \circ [SX, Z] + [SY, W] \circ Sf\big)$$

*Proof.* See [Toda, Lemma 1.1]. □

**Proposition 2.2.** *For a sequence of maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

*we have*

*(i)* $\{k, h, g\} \circ f \subseteq \{k, h, g \circ f\}$
*(ii)* $\{k, h, g \circ f\} \subseteq \{k, h \circ g, f\}$
*(iii)* $\{k \circ h, g, f\} \subseteq \{k, h \circ g, f\}$
*(iv)* $k \circ \{h, g, f\} \subseteq \{k \circ h, g, f\}$

*Proof.* See [Toda, Proposition 1.2]. □

**Proposition 2.3.** *Let*

$$K(\mathbb{Z},n) \xrightarrow{p} K(\mathbb{Z},n) \xrightarrow{\rho} K(\mathbb{F}_p,n) \xrightarrow{\beta} K(\mathbb{Z},n+1)$$

*represent multiplication by $p$, reduction mod $p$, and the mod $p$ Bockstein, respectively. Then*

$$\mathrm{id} \in \{\beta,\rho,p\} \subseteq [SK(\mathbb{Z},n), K(\mathbb{Z},n+1)] = [K(\mathbb{Z},n), K(\mathbb{Z},n)]$$

□

**Corollary 2.4.** *Let $c : X \to K(\mathbb{Z},n)$ represent a cohomology class. Let $\rho$ and $\beta$ be as in Proposition 2.3. Then*

$$\{\beta,\rho,c\} = \tfrac{1}{p}c + \mathbb{Z}c \subseteq H^n(X) = [SX, K(\mathbb{Z},n+1)]$$

*where*

$$\tfrac{1}{p}c = \{c' | pc' = c\}$$

*Proof.* Clearly the two sides have the same indeterminacy $\mathbb{Z}c + \beta H^{n-1}(X;\mathbb{F}_p)$, so all we need to check is that if $pc' = c$, then $c' \in \{\beta,\rho,c\}$. But this follows from Proposition 2.3:

$$\{\beta,\rho,p\circ c'\} \supseteq \{\beta,\rho,p\}\circ c' \ni c'$$

□

## 3. Elementary properties of the secondary classes

Consider the oriented Grassmannian $\mathrm{SO}(N+2)/\mathrm{SO}(N)\times\mathrm{SO}(2)$. Let $U = U_N = \mathrm{SO}(N+2)\times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^2$ be the canonical oriented 2-dimensional vectorbundle and let $U^\perp = U_N^\perp = \mathrm{SO}(N+2)\times_{\mathrm{SO}(N)\times\mathrm{SO}(2)} \mathbb{R}^N$ be its orthogonal complement.

**Lemma 3.1** ([GMT]). *In $H^*(\mathrm{Th}(U^\perp),*;\mathbb{F}_p)$ we have that*

$$\mathcal{P}^i\lambda_{U^\perp} = (-1)^i e^{i(p-1)}\lambda_{U^\perp}$$

*Proof.* Let $\mathcal{P} = \sum_i \mathcal{P}^i$. Then $\mathcal{P}(\lambda_U) = (1+e(U)^{p-1})\lambda_U$. Since $\lambda_{U\oplus U^\perp} = \lambda_U \cup \lambda_{U^\perp}$ we get

$$\lambda_U\cup\lambda_{U^\perp} = \lambda_{U\oplus U^\perp} = \mathcal{P}(\lambda_{U\oplus U^\perp}) = \mathcal{P}(\lambda_U)\cup\mathcal{P}(\lambda_{U^\perp}) = (1+e(U)^{p-1})\lambda_U\cup\mathcal{P}(\lambda_{U^\perp})$$

and hence

$$\mathcal{P}(\lambda_{U^\perp}) = (1+e(U)^{p-1})^{-1}\lambda_{U^\perp} = \left(\sum_i (-1)^i e(U)^{i(p-1)}\right)\lambda_{U^\perp}$$

□

*Proof of Lemma 1.1.* Clearly $l\circ\alpha \simeq 0$. The cohomology of the Grassmannian $\mathrm{SO}(N+2)/\mathrm{SO}(N)\times\mathrm{SO}(2)$ vanishes in odd degrees (when $N$ is larger than the degree), so $\beta\mathcal{P}^i\circ\lambda \simeq 0$. Therefore $\{\beta\mathcal{P}^i,\lambda,\alpha\}$ is defined. It follows from Lemma 2.1 that the indeterminacy is $\mathbb{Z}\kappa_{i(p-1)-1}$. □

*Proof of Theorem 1.3.* This follows from Proposition 2.2 and Corollary 2.4 and the diagram:

$$\begin{tikzcd}
B_+ \wedge S^{N+2} \arrow[r, "\alpha"] \arrow[dr, "\kappa_{i(p-1)-1}"'] & \mathrm{Th}(U_N^\perp) \arrow[rr, "\lambda"] \arrow[d, "e^{i(p-1)}\lambda"] & & K(\mathbb{Z}, N) \arrow[d, "\mathcal{P}^i"] \\
& K(\mathbb{Z}, N+2i(p-1)) \arrow[rr, "\rho"] & & K(\mathbb{F}_p, N+2i(p-1)) \arrow[d, "\beta"] \\
& & & K(\mathbb{Z}, N+2i(p-1)+1)
\end{tikzcd}$$

Indeed, Proposition 2.2 gives the inclusions

$$\begin{aligned}\{\beta, \rho, \kappa_{i(p-1)-1}\} &= \{\beta, \rho, (e^{i(p-1)}\lambda) \circ \alpha\} \subseteq \{\beta, \rho \circ (e^{i(p-1)}\lambda), \alpha\} \\ &= (-1)^i\{\beta, \mathcal{P}^i\lambda, \alpha\} \supseteq (-1)^i\{\beta\mathcal{P}^i, \lambda, \alpha\} = \lambda_i(E).\end{aligned}$$

Then Lemma 2.1 proves that the first inclusion is an equality since the two sides have the same indeterminacy $\mathrm{Im}(\beta) + \mathbb{Z}\kappa_{i(p-1)-1}$. Therefore by Corollary 2.4

$$\lambda_i(E) \subseteq \{\beta, \rho, \kappa_{i(p-1)-1}\} = \tfrac{1}{p}\kappa_{i(p-1)-1} + \mathbb{Z}\kappa_{i(p-1)-1},$$

and hence

$$p\lambda_i(E) \subseteq (1 + p\mathbb{Z})\kappa_{i(p-1)-1}.$$

Since they have the same indeterminacy, they are equal. $\square$

*Proof of Theorem 1.4.* *(i)* follows from the additivity of $\alpha$, i.e. the property that $\alpha(E \amalg E') = \alpha(E) + \alpha(E') \in [B_+ \wedge S^{N+2}, \mathrm{Th}(U_N^\perp)]$. Similarly *(ii)* follows from the property that $\alpha(E) = \alpha(E')$ when $E'$ is obtained from $E$ by fibrewise surgery. *(iii)* follows from *(i)* and *(ii)* since $E \cup_{S^1\times B} E'$ is obtained from $\big(E \cup_{S^1\times B} (D^2 \times B)\big) \amalg \big(E \cup_{S^1\times B} (D^2 \times B)\big)$ by fibrewise surgery. $\square$

## 4. A variant of $\lambda_{ps}$

The goal of this section is to prove Theorem 1.5. The definition and properties of $\lambda_i$ proves that $\kappa_{i(p-1)}$ is divisible by $p$. When $i = ps$, a variant of $\lambda_{ps}$ can be used to prove that $\kappa_{ps(p-1)-1}$ is divisible by $p^2$.

**Definition 4.1.** Let $s \geq 0$ and consider the Steenrod algebra $\mathscr{A}_p$. When $p = 2$ we write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$ as before. Define $\theta_s \in \mathscr{A}_p$ by

$$\theta_s = \sum_{j=0}^{s}(-1)^j\binom{(p-1)(s-j)}{j}\mathcal{P}^{ps-j}\mathcal{P}^j = \mathcal{P}^{ps} + \text{terms of lenght 2}$$

Define vectors $v_s, w_s \in \mathscr{A}_p$ by

$$w_s = (\mathcal{P}^0, \dots \mathcal{P}^s), \quad v_s = (\mathcal{P}^{ps}, \dots, (-1)^j\binom{(p-1)(s-j)-1}{j}\mathcal{P}^{ps-j}, \dots, \mathcal{P}^{(p-1)s}).$$

**Lemma 4.2.** *(i) In $H^*(\mathrm{Th}(U^\perp), *; \mathbb{F}_p)$ we have that $\theta_s \lambda_{U^\perp} = e^{ps(p-1)} \lambda_{U^\perp}$.*
*(ii) $v_s^T \beta w_s = \beta\theta_s$.*

*Proof.* *(i)* This is similar to Lemma 3.1, using the fact that the admissible terms of length 2 act trivially on $\lambda_{U^\perp}$. Formula *(ii)* is the Adem relation for $\mathcal{P}^{(p-1)s}\beta\mathcal{P}^s$. □

**Definition 4.3.** Let $\alpha, \lambda, \theta_s$ be as above. Define the secondary characteristic class

$$\tilde{\lambda}_{ps} = (-1)^s \{\beta\theta_s, \lambda, \alpha\} \in H^{2ps(p-1)-2}(B, \mathbb{Z})/\mathbb{Z}\kappa_{ps(p-1)-1}$$

Notice that $\tilde{\lambda}_{ps}$ satisfies the same formal properties as $\lambda_{ps}$. In particular $p\tilde{\lambda}_{ps} = (1 + p\mathbb{Z})\kappa_{ps(p-1)-1}$. In general $\tilde{\lambda}_{ps} \neq \lambda_{ps}$.

*Proof of Theorem 1.5.* We have

$$\begin{aligned}(-1)^s \rho \circ \{\beta\theta_s, \lambda, \alpha\} \subseteq (-1)^s \{\rho \circ \beta\theta_s, \lambda, \alpha\} &= (-1)^s \{v_s^T \beta w_s, \lambda, \alpha\} \\ &\supseteq (-1)^s v_s^T \{\beta w_s, \lambda, \alpha\}\end{aligned}$$

and it is seen that all the inclusions are equalities since the indeterminacy vanishes. Since

$$(-1)^s \{\beta w_s, \lambda, \alpha\} \in \prod_{i=0}^{s} H^{N+2i(p-1)}(B_+ \wedge S^{N+2}; \mathbb{F}_p) = \prod_{i=0}^{s} H^{2i(p-1)-2}(B; \mathbb{F}_p),$$

$v^T$ will vanish since $H^*(B; \mathbb{F}_p)$ is an unstable $\mathscr{A}_p$-module.

Hence the mod $p$ reduction of $\tilde{\lambda}_{ps}$ vanishes, so $\kappa_{ps(p-1)-1} = p\tilde{\lambda}_{ps} = 0 \in H^*(B; \mathbb{Z}/p^2)$. □

## References

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